Remark:

1. We can write

$$X(t+\tau) = X(t+\tau) - X(t) + X(t) = Y_\tau + X(t)$$

   Thus $X(t)$ can be viewed as a random walk on a line. After a time step τ, the particle's position has moved by an amount Y_τ that is independent of the previous position $X(t)$.

2. The pdf of Y_τ is given by

$$f_{Y_\tau}(x) = \frac{1}{\sqrt{2\pi\alpha\tau}} e^{-\frac{x^2}{2\alpha\tau}}, -\infty < x < \infty$$

The Brownian Motion Process

R. Brown (1827, a botanist) studied the motion of pollen particles suspended in water and lent him name to the type of erratic movements.

Note. A Poisson process is a continuous time and discrete value process. A Brownian process is a continuous time and continuous value process.

Think about that $X(t)$ is a position of a particle at time instance t, and $X(t)$ may not be nondecreasing with time.

Definition. $X(t)$ is a Brownian motion process if

- $X(0) = 0$;
- for $\tau > 0$, $Y_\tau = X(t+\tau) - X(t)$ is a normal (Gaussian) random variable with mean 0 and variance $\alpha\tau$;
- Y_τ is independent of $X(t')$ for all $t' \leq t$.

Autocovariance and Autocorrelation

Recall that covariance of two random variables X and Y indicate that how much information random variable X (or Y) provides about Y (or X).

$$\rho(X(t), X(t+\tau)) = \frac{Cov(X(t), X(t+\tau))}{\sqrt{Var(X(t))Var(X(t+\tau))}}$$

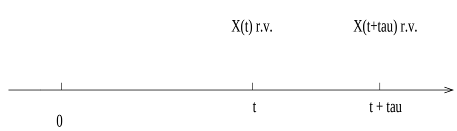
$\lvert\rho(X(t), X(t+\tau))\rvert$	$\lvert Cov(X(t), X(t+\tau))\rvert$	Relation
$\longrightarrow 1$	increases	an observation of $X(t)$ provides an accurate indication of the values of $X(t+\tau)$ (since they tend to have a linear relation)
$\longrightarrow 0$	decreases	$X(t+\tau)$, rapidly changed from $X(t)$

Expected Values, Autocovariance and Autocorrelation

Definition 1 *The expected value (or mean) of a stochastic process $X(t)$ is the expected value of the r.v. $X(t)$:*

$$\mu_X(t) = E[X(t)].$$

It is, in general, a function of time.



Example 1 *We continue the Example in Sec. 1. At the receiver of an AM radio, the received signal contains a cosine carrier signal at the carrier frequency f_c with a random phase Θ that is uniformly distributed over $[-\pi, \pi]$. The received signal is*

$$X(t) = Acos(2\pi f_c t + \Theta)$$

Find the expectation and the autocorrelation of $X(t)$.

Definition 2 *The autocovariance $Cov_X(t, \tau)$ of $X(t)$ is the covariance of the r.v. $X(t)$ and $X(t+\tau)$:*

$$C_X(t, \tau) = Cov_X(X(t), X(t+\tau)).$$

Definition 3 *The autocorrelation $R_X(t, \tau)$ of a process $X(t)$ is the joint moment of the r.v. $X(t)$ and $X(t+\tau)$:*

$$R_X(t, \tau) = E[X(t), X(t+\tau)].$$

Theorem 1 *The autocovariance and autocorrelation of $X(t)$ are related as follows*

$$C_X(t, \tau) = R_X(t, \tau) - \mu_X(t)\mu_X(t+\tau)$$

Remark. If $\tau = 0$, then $C_X(t, 0) = Var(X(t))$. In other words, the variance of the r.v. $X(t)$ is given by

$$Var(X(t)) = C_X(t, 0) = R_X(t, 0) - (\mu_X(t))^2$$